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DISAPPEARING INTERFACES IN NONLINEAR DIFFUSION

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Abstract

We study the large-time behaviour and the behaviour of the interfaces of the nonlinear diffusion equation

$$\rho(x)u_t = \Delta A(u)$$

in one and two space dimensions. The function A is of porous media type, smooth but with a vanishing derivative at some values of u, and $\rho > 0$ is supposed continuous and bounded from above. If ρ is not bounded away from zero, the large-time behaviour of solutions and their interfaces can be essentially different from the case when ρ is constant. We extend results by Rosenau and Kamin [13] and derive the large-time asymptotic behaviour of solutions, as well as a precise characterisation of the behaviour of the interfaces of solutions in one space dimension and in some cases in two space dimensions. In one space dimension and when ρ is monotonic the result states that the interface $\zeta(t) = \sup\{x \in \mathbb{R} : u(x,t) > 0\}$ tends to infinity in finite time if and only if $\int_0^\infty x\rho(x)\,dx < \infty$.

1 Introduction

In this article we study some properties of solutions of the nonlinear diffusion equation

$$\rho(x)u_t = \Delta A(u) \qquad x \in \mathbb{R}^N,\ t > 0, \tag{1.1}$$

in one and two space dimensions. The nonlinearity A is such that $A' > 0$ on $(0,1)$ and $A'(0) = A'(1) = 0$; the density function $\rho : \mathbb{R}^N \to (0,\infty)$ is supposed bounded and continuous, and we shall mostly be interested in the case where $\rho(x)$ tends to zero for large $|x|$.

Equations of type (1.1) arise in plasma physics [10, 13], and in hydrology [8, 2, 7], and in order to set the ideas we shall briefly describe the hydrological model. In the interaction between fresh and salt water in underground aquifers, mixing of the two liquids occurs over length scales much smaller than the size of the aquifer, and in modelling this situation it is therefore generally assumed that a sharp interface separates the liquids. In a horizontal aquifer of even thickness, and under the assumption that the slope of the interface is not too large, the movement of the interface is governed by the equation [8, 2]

$$\varepsilon(x,y)\frac{\mu}{\gamma}\frac{\partial u}{\partial t} - \operatorname{div}\left(\kappa(x,y)\,u(1-u)\frac{\nabla u}{1+|\nabla u|^2}\right) = 0. \tag{1.2}$$

Here $u(x,y)$ represents the height of the interface, scaled to take values between zero and one. The constants μ and γ represent the viscosity and the density difference between the fluids, ε is the porosity, and κ is the permeability of the medium.

Since we shall mainly be interested in solutions u with relatively small gradients, we replace the quotient $\nabla u/(1+|\nabla u|^2)$ in (1.2) by ∇u. Furthermore, we shall mostly consider either one-dimensional or two-dimensional axially symmetric solutions. In the two-dimensional case with axial symmetry, equation (1.2) reduces to

$$\varepsilon(r)\frac{\mu}{\gamma}u_t - \frac{1}{r}\left(r\kappa(r)\,u(1-u)u_r\right)_r = 0 \tag{1.3}$$

where $r^2 = x^2+y^2$ and subscripts denote differentiation. If we introduce a new space variable $\tilde r$, defined by

$$\log\tilde r := \int_1^r \frac{ds}{s\kappa(s)}$$

then (1.3) transforms into

$$\rho(\tilde r)u_t - \frac{1}{\tilde r}(\tilde r u(1-u)u_{\tilde r})_{\tilde r} = 0 \tag{1.4}$$

in which $\tilde r^2\rho(\tilde r) = (\mu/\gamma)\,r^2\varepsilon(r)\kappa(r)$. In one space dimension, the equation becomes

$$\rho(x)u_t - \left(u(1-u)u_x\right)_x = 0. \tag{1.5}$$

Both (1.4) and (1.5) are of the form (1.1).

We shall suppose that the degeneration of the nonlinearity A is such that at the values $u=0$ and $u=1$ interfaces can appear (we shall henceforth use the term 'interfaces' in the mathematical sense that is common in degenerate diffusion, instead of the physical sense used above). Such is the case for equations (1.4) and (1.5) above. Our main interest in this paper lies in the behaviour of solutions of (1.1) and their interfaces for large time. This interest was fired by previous works by Kamin and Rosenau [10, 13] on equation (1.1) with single degeneration ($A'(0)=0$, $A'(s)>0$ for all $s>0$). Among other results they showed that as time tends to infinity the solution u converges uniformly on bounded sets to the weighted mean of the initial distribution u_0, i.e. $u\to\bar u$ where $\bar u$ is given by

$$\bar u := \frac{\int\rho(x)u_0(x)\,dx}{\int\rho(x)\,dx},$$

provided the numerator of this expression has a finite value. This extends a known result in the case of constant ρ, which states that a solution with finite initial mass decays to zero.

Recently an interesting result has been proved by Kamin and Kersner in [9]. They consider equation (1.1) in $\mathbb{R}^N$ with $N\geq 3$, again with single degeneration, and they proved that integrability of ρ on $\mathbb{R}^N$ ($\rho \in L^1(\mathbb{R}^N)$) implies that even if the initial distribution has compact support and therefore the solution also has compact support for small times, there is a time $0 < T < \infty$ such that for $t > T$ the support is no longer compact. This behaviour differs strongly from the case of constant ρ, in which the support of the solution is a compact set for all time $t > 0$. For the same equation a converse result has been proved in [12]: in this paper the author exhibits an explicit supersolution that also has compact support for small time; in the case that ρ is radially symmetric and decreasing in r, the support of this supersolution remains bounded for all time if and only if $r\rho(r) \notin L^1(0,\infty)$. By means of the comparison principle this implies that if $r\rho(r) \notin L^1(0,\infty)$, then a solution of (1.1) with bounded initial support has a bounded support at all finite time.

In this article we shall be interested in the Cauchy problem for (1.1) in one and two space dimensions. This dimensional restriction is natural in the case of the hydrological model, and also the mathematical properties that we wish to examine are different for dimensions one and two on one hand and three and higher on the other. Since we will be interested in solutions with interfaces between the regions $\{u = 0\}$, $\{0 < u < 1\}$, and $\{u = 1\}$, we assume that

$$\mathrm{H}_A \quad \begin{cases} A \in C^1([0,1]),\ A' > 0 \text{ on } (0,1),\ A(0) = A'(0) = A'(1) = 0, \\ \displaystyle\int_{0+} \frac{A'(s)}{s}\,ds < \infty \quad \text{and} \quad \int^{1-} \frac{A'(s)}{1-s}\,ds < \infty. \end{cases}$$

In addition, the density function ρ and the initial data u_0 should satisfy

$$\mathrm{H}_\rho \quad \rho \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ \rho > 0 \text{ on } \mathbb{R}^N;$$
$$\mathrm{H}_0 \quad u_0 \in C(\mathbb{R}^N),\ 0 \le u_0 \le 1 \text{ on } \mathbb{R}^N.$$

Throughout this article we shall suppose that these hypotheses are satisfied.

To our knowledge, existence and uniqueness for the Cauchy problem associated with (1.1) have not yet been proved in the literature. We therefore include these proofs in the Appendix. The uniqueness is a consequence of the following Comparison Principle:

Theorem 1.1 *Let N be equal to either one or two, and suppose that u_1 is a subsolution and u_2 a supersolution of Problem (P). If $\rho(u_{01}-u_{02})_+ \in L^1(\mathbb{R}^N)$, then $\rho(u_1-u_2)_+(\cdot,t) \in L^1(\mathbb{R}^N)$ for all $t \ge 0$ and*

$$\int_{\mathbb{R}^N} \rho(u_1 - u_2)_+(\cdot,t) \le \int_{\mathbb{R}^N} \rho(u_{01} - u_{02})_+$$

for all $t \ge 0$.

The definition of sub- and supersolutions is given in the Appendix.

We prove the following theorems.

Theorem 1.2 (Large-time behaviour) *Let N be equal to either one or two, and let u be the solution of (1.1) with initial data u_0. If $\rho u_0 \in L^1(\mathbb{R}^N)$, then*

$$u(t) \to \bar{u} := \frac{\displaystyle\int_{\mathbb{R}^N} \rho(x) u_0(x)\,dx}{\displaystyle\int_{\mathbb{R}^N} \rho(x)\,dx} \quad as \quad t \to \infty,$$

as $t \to \infty$, uniformly on compact subsets of $\mathbb{R}^N$.

Eidus has remarked in [5] that a similar result holds in the case of a single degeneration in two space dimensions.

Let the support of a function f ($\operatorname{supp} f$) be defined as the closure of the set $\{x : f(x) > 0\}$. A solution u of (1.1) for $N = 1$ is said to exhibit finite time blow-up if its support is bounded from above initially and there exists a time T such that $\operatorname{supp} u(t)$ is unbounded from above for all time $t > T$. For the formulation of Theorem 1.3 we shall need an auxiliary density function σ defined by

$$\sigma(x) = \min_{0 \le \xi \le x} \rho(\xi),$$

the reason being that the function σ is monotonic while ρ need not be.

Theorem 1.3 (Blow-up in one dimension) *Let u be a solution of (1.1) for $N = 1$, with non-zero initial data u_0, such that the support of u is bounded from above at time $t = 0$. Then the following implications hold:*

(i) $\displaystyle\int_0^\infty x\rho(x)\,dx < \infty \implies$ *finite time blow-up;*

(ii) $\displaystyle\int_0^\infty x\sigma(x)\,dx = \infty \implies$ *no finite time blow-up.*

If ρ is not decreasing, the two conditions above leave a small gap. In the class of decreasing functions ρ, however, the characterisation is complete:

Corollary 1.4 *Let the conditions of Theorem 1.3 be satisfied, and suppose in addition that ρ is non-increasing on $[K, \infty)$ for some $K > 0$. Then*

$$\text{finite time blow-up} \iff \int_0^\infty x\rho(x)\,dx < \infty.$$

It follows from the inversion $\tilde{u} = 1 - u$ that similar statements hold for the interface at $u = 1$. Note that the behaviour of ρ and u_0 towards $-\infty$ has no influence on the (qualitative) behaviour of the upper boundary of the support. We can apply these statements once to $\{x > 0\}$ and once to $\{x < 0\}$ with independent results.

Using the Comparison Principle we can extend this result to a statement on a strip $\Omega = \mathbb{R} \times (-1, 1)$ with Neumann boundary conditions, with a density function ρ that does not depend on the vertical coordinate: $\rho(x, y) = \rho(x)$ on Ω. Consider the problem

$$(1.6) \qquad \begin{cases} \rho u_t = \Delta A(u) & \text{in } Q_T = \Omega \times (0, T] \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T] \\ u = u_0 & \text{at } t = 0. \end{cases}$$

The following result easily follows from the Comparison Prinicple:

Theorem 1.5 (Blow-up in a 2d strip) *Let the initial condition u_0 be such that $u_0(x,y) = 1$ for small x and $u_0(x,y) = 0$ for large x. Let $\zeta_0(t)$ denote the interface between $\{u > 0\}$ and $\{u = 0\}$ at time t:*

$$\zeta_0(t) = \operatorname{supp} u(t) \cap \{(x,y) \in \Omega : u(x,y,t) = 0\}.$$

Then the following statements hold:

(i) If $\int_0^\infty x\rho(x)\,dx < \infty$ then the interface ζ_0 will run off to infinity in finite time;

(ii) If $\int_0^\infty x\sigma(x)\,dx = \infty$ then the interface ζ_0 will remain bounded for all finite time.

A similar statement holds for the interface ζ_1 between the sets $\{u = 1\}$ and $\{u < 1\}$.

A different way of extrapolating the one-dimensional results is by considering the two-dimensional radially symmetric problem and transforming the ensuing (one-dimensional) equation to an equation of the form (1.1). In this case the auxiliary density function σ is different:

$$\sigma(r) = \min_{0 \le \xi \le r} \xi^2 \rho(\xi).$$

We prove the following result:

Theorem 1.6 (Blow-up in 2d, radially symmetric case) *Let u be a solution of (1.1) with initial condition u_0. Suppose that both ρ and u_0 are radially symmetric, and that $\operatorname{supp} u_0$ is compact.*

(i) If $\int_1^\infty \rho(r) r \log r\,dr < \infty$ and $0 \in \operatorname{Int}(\operatorname{supp} u_0)$, then the support of u ceases to be compact in finite time;

(ii) If $\int_1^\infty \sigma(r)\frac{\log r}{r}\,dr = \infty$, then the support of u is compact for all time.

Corollary 1.7 *Suppose u_0 has compact support and $0 \in \operatorname{Int}(\operatorname{supp} u_0)$. If $r \mapsto r^2\rho(r)$ is a decreasing function of r on a neighbourhood of $+\infty$, then the support of u becomes unbounded in finite time if and only if*

$$\int_1^\infty \rho(r) r \log r\,dr < \infty.$$

Remark 1.1. The proof of part (ii) of Theorem 1.3 is based on the construction of a supersolution. This construction can be done in all dimensions $N \ge 1$ [12], leading to the following theorem:

Theorem 1.8 *Let $N \ge 1$ and define $\sigma(r) = \min\{\rho(x) : |x| \le r\}$ for $0 \le r < \infty$. Suppose the solution u of Problem (P) has compact support initially. If*

$$\int_0^\infty r\sigma(r)\,dr = \infty$$

then $\operatorname{supp} u(t)$ will be bounded for all time $t \ge 0$.

There is an interesting gap between the statements of Theorem 1.8 for $N = 2$ and Corollary 1.7. Clearly, the condition $r\sigma(r) \notin L^1(0,\infty)$ is too weak in the case of radially symmetric densities. But if we take a density function $\rho = \rho(x,y)$ on $\mathbb{R}^2$ that is only a function of x, i.e. $\rho(x,y) = \rho(x)$, then in the same way as in Theorem 1.5 we can compare it with solutions of the one-dimensional problem. The result of this comparison is that for convenient initial distributions the blow-up of interfaces is *equivalent* with $x\rho(x) \in L^1(0,\infty)$, which implies that the condition $r\sigma(r) \notin L^1(0,\infty)$ is sharp. It is not clear what a general condition for blow-up of interfaces should be in a non-radially symmetric situation.

Theorem 1.2 is proved in Section 2. The blow-up of interfaces in one space dimension (Theorem 1.3) is studied in Section 3, and in two space dimensions in Section 4 (Theorem 1.6). Shortly before submitting the manuscript of this article to the editor, a manuscript by V. A. Galaktionov, S. Kamin, and R. Kersner was brought to our attention, in which Theorem 1.3 is also proved under more restrictive conditions on ρ.

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2 Proof of Theorem 1.2

Theorem 1.2 was proved for the single-degeneration case in one dimension by Rosenau and Kamin [13]. We give here a completely different proof which also applies to the case studied by Rosenau and Kamin.

We shall use certain a priori estimates on the solution of Problem (P). The following Lemma is proved in Appendix A:

Lemma 2.1 *Let u be the solution of Problem (P) with initial function u_0, and set $v = A(u)$. Suppose that $\rho u_0 \in L^1(\mathbb{R}^N)$. Then the following statements hold.*

$$\text{(i)} \quad \int_{\mathbb{R}^N} \rho u(\cdot,\tau) = \int_{\mathbb{R}^N} \rho u_0 \quad \text{for all} \quad \tau \geq 0 \quad \text{(conservation of mass);}$$

$$\text{(ii)} \quad \int_{\mathbb{R}^N} \rho B(v(\cdot,\tau)) + \int_0^\tau \int_{\mathbb{R}^N} |\nabla v|^2 \leq \int_{\mathbb{R}^N} \rho B(v_0) \quad \text{for all} \quad \tau \geq 0;$$

$$\text{(iii)} \quad \int_{\mathbb{R}^N} |\nabla v|^2(\cdot,\tau) \leq \frac{c}{\tau} \quad \text{for all} \quad \tau > 0.$$

where $B(s) = \int_0^s \sigma\beta'(\sigma)d\sigma$ with $\beta = A^{-1}$ and $c > 0$ is a constant that does not depend on τ.

Remark 2.1. Estimates as given in Lemma 2.1 are well known in degenerate diffusion problems. It should be noted, however, that the conservation of mass is only true in one and two space dimensions; indeed, the main result in [9], which holds for $N \geq 3$ (see the Introduction), is proved by obtaining a contradiction to this statement.

Proof of Theorem 1.2. It follows from the uniform continuity of the function v (this is a consequence of [4], as is shown in the proof of Theorem A.2) that there exists a sequence $t_n \to \infty$ and a function $\bar{v} \in C(\mathbb{R}^N)$, $0 \le \bar{v} \le A(1)$, such that $v(t_n) \to \bar{v}$ as $n \to \infty$, uniformly on compact sets. Now let Ω be an arbitrary bounded set of $\mathbb{R}^N$. Then by Lemma 2.1, part (iii),

$$\left\| v(t_n) - \frac{1}{|\Omega|}\int_\Omega v(t_n) \right\|_{L^2(\Omega)} \le C\|\nabla v(t_n)\|_{L^2(\Omega)} \le \frac{cC}{t_n},$$

where C is a constant that depends on Ω, so that

$$\bar{v} = \frac{1}{|\Omega|}\int_\Omega \bar{v}$$

for each bounded subset $\Omega \subset \mathbb{R}^N$. Therefore $\bar{v}$ is constant, and $u(t_n) = \beta(v(t_n)) \to \bar{u} := \beta(\bar{v})$ as $t_n \to \infty$, where $\beta := A^{-1}$. The value of $\bar{u}$ follows from the conservation of mass (part (i) of Lemma 2.1). The fact that this limit is uniquely defined implies the convergence of $u(t)$ as $t \to \infty$. This concludes the proof of Theorem 1.2. •

3 Proof of Theorem 1.3

The proof of Theorem 1.3 is based on the comparison principle. First we consider a special case.

Lemma 3.1 *Let $u_0 \in L^1(\mathbb{R})$, $u_0 \not\equiv 0$, and suppose that the support of u_0 is bounded from above. If $\int_0^\infty x\rho(x)\,dx$ is finite, then there exists a time T after which the support of the solution u is unbounded from above.*

Proof. Define the upper interface function

$$\zeta(t) = \sup\{x \in \mathbb{R} : u(x,t) > 0\}.$$

For the purpose of contradiction we suppose that $\zeta(t) < \infty$ for all $t \in [0,\infty)$. Let the sequence of smooth functions χ_n be such that $\operatorname{supp}\chi_n$ is compact in $(0,\infty)$, χ_n and $|x\chi_n'(x)|$ are bounded uniformly in x and n, and finally $\chi_n \to 1$ and $\chi_n' \to 0$ pointwise on $(0,\infty)$. We substitute the test function

$$\psi(x) = \begin{cases} x\chi_n(x) & \text{if } x > 0, \\ 0 & \text{if } x \le 0 \end{cases}$$

in equation (A.1). Then

$$\begin{aligned}\int_0^\infty x\rho(x)u(x,T)\chi_n(x)\,dx - \int_0^\infty x\rho(x)u_0(x)\chi_n(x)\,dx &= \int_0^T\int_0^\infty A(u)\{x\chi_n\}_{xx}\,dxdt \\ &= -\int_0^T\int_0^\infty A(u)_x\{\chi_n + x\chi_n'\}\,dxdt.\end{aligned}$$

Note that the function $A(u)_x$ is well-defined by Lemma 2.1. Letting $n \to \infty$ and applying Lebesgue's dominated convergence theorem we deduce that

$$(3.1)\qquad \int_0^\infty x\rho(x)u(x,T)\,dx - \int_0^\infty x\rho(x)u_0(x)\,dx = -\int_0^T\int_0^{\zeta(t)} A(u)_x\,dxdt = \int_0^T A(u(0,t))\,dt.$$

If $\rho \in L^1(\mathbb{R})$, then by Theorem 1.2, $u(0,t) \to \bar{u} > 0$ as $t \to \infty$. Since the left-hand side of (3.1) is bounded as $T \to \infty$, there exists a sequence $\{t_n\}$, $\lim_{n\to\infty} t_n = \infty$, such that $A(u(0,t_n)) \to 0$ as $n \to \infty$, implying a contradiction. On the other hand, if $\rho \notin L^1(\mathbb{R})$, then by Theorem 1.2 the function $u(\cdot,t)$ converges to zero pointwise on $\mathbb{R}$ as $t \to \infty$. By the dominated convergence theorem we conclude that the first integral in (3.1) tends to zero as $T \to \infty$. At some time T there will be a sign difference between the left and the right hand side of (3.1), again implying a contradiction. •

We now turn to the proof of Theorem 1.3. First consider the case in which $\int_0^\infty x\rho(x)\,dx < \infty$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function such that $\chi(x) = 1$ for all $x > 0$, $\chi(x) = 0$ for all $x < -1$ and $0 \le \chi \le 1$ on $\mathbb{R}$. Define $v_0(x) = u_0(x)\chi(x+d)$ for such a value of $d > 0$ that v_0 is not identically equal to zero. Then $v_0 \in L^1(\mathbb{R})$, and $\operatorname{supp} v_0$ is bounded from above. If we denote the solution of Problem (P) with initial data v_0 by v, then Lemma 3.1 implies that $\operatorname{supp} v$ will be unbounded from above in finite time. Since by the comparison principle $u \ge v$ on $\mathbb{R}\times\mathbb{R}^+$, the same holds for u.

Now assume that $\int_0^\infty x\sigma(x)\,dx = \infty$. In order to show that the support of u remains bounded for all time, we compare the solution u with a supersolution with bounded support. A similar supersolution was discussed in [12].

Suppose for the time being that $u_0(x) = 0$ for all $x \ge 0$. Let the comparison function w be defined by

$$w(x,t) = \begin{cases} 1 & x \le 0 \\ \eta^{-1}\left[a\left(1 - x^2/g(t)^2\right)\right] & 0 < x < g(t) \\ 0 & x \ge g(t), \end{cases}$$

where $\eta(s) = \int_0^s A'(\tau)/\tau\,d\tau$, $\eta(1) = a$, and $g : [0,\infty) \to [0,\infty)$ is a function to be specified later. By explicit calculation it follows that the following conditions are sufficient to guarantee that w is a weak supersolution in the sense of Definition A.1:

$$(3.2)\qquad \rho w_t \ge A(w)_{xx} \qquad \text{for } 0 < x < g(t),\ t > 0$$

$$(3.3)\qquad g'(t) \ge -\frac{1}{\rho(g(t))}\frac{\partial}{\partial x}\eta(w)(g(t),t) \qquad \text{for all } t > 0$$

$$(3.4)\qquad w(x,0) \ge u_0(x) \qquad \text{for all } x \in \mathbb{R}.$$

This follows from the following argument: if $P = \{(x,t) \in \mathbb{R}\times\mathbb{R}^+ : |x| < g(t)\}$ and $\Gamma = \{(x,t) \in \mathbb{R}\times\mathbb{R}^+ : |x| = g(t)\}$, then it follows from (A.1) that w is a supersolution if

$$(3.5)\qquad -\int_P \{\rho w_t - A(w)_{xx}\}\psi + \int_\Gamma \{\rho w\nu_t - A(w)_x\nu_x\}\psi \le 0$$

for all appropriate test functions ψ. Here $\nu = (\nu_t, \nu_x)$ is the unit vector normal to P that points outward. If g is differentiable then $\nu_t = -g'(t)\nu_x$, and by conditions (3.2) and (3.3), condition (3.5) is met. The condition (3.4) is necessary to apply the comparison principle (Theorem 1.1).

Inequality (3.4) is satisfied due to our assumption that the support of u_0 is contained in $\{x \leq 0\}$. If we expand (3.2) we find

$$x^2\left\{\frac{\rho(x)g'(t)}{g(t)} - \frac{2a}{g(t)^2}\right\} \geq -A'(w) \quad \text{for} \quad 0 < x < g(t),\ t > 0. \tag{3.6}$$

The right-hand side is non-positive and therefore it is sufficient to require that g satisfy

$$g'(t) \geq \frac{2a}{g(t)\rho(x)} \quad \text{for all} \quad 0 < x < g(t),\ t > 0.$$

With the definition of σ in mind we define g by setting

$$\begin{cases} g'(t) = \dfrac{2a}{g(t)\sigma(g(t))} & \text{for all} \quad t > 0 \\ g(0) = 1. \end{cases} \tag{3.7}$$

Since $\partial\eta(w)/\partial x$ takes the value $-2a/g(t)$ in $x = g(t)$, with this definition of g the function w also satisfies (3.3).

Now that the comparison function has been defined, we need to determine the behaviour of its interface $\{(x,t) : x = g(t)\}$. The solution g of the problem (3.7) is given by

$$\int_1^{g(t)} x\sigma(x)\,dx = 2at. \tag{3.8}$$

From the initial assumption $x\sigma(x) \notin L^1(0,\infty)$ it follows that $g(t)$ remains finite for all finite time t. By the comparison principle the same holds for u.

We can relax the condition on the support of u_0 by shifting the supersolution rightwards until the initial distributions u_0 and $w(\cdot, 0)$ are ordered. If w is shifted rightwards by a distance $d > 0$, then the ensuing condition on the behaviour of σ is $\int_d^\infty (x-d)\sigma(x)\,dx = \infty$; since

$$\int_d^\infty (x-d)\sigma(x)\,dx \geq \int_d^{2d} (x-d)\sigma(x)\,dx + \tfrac{1}{2}\int_{2d}^\infty x\sigma(x)\,dx = \infty,$$

this condition is satisfied. This concludes the proof of Theorem 1.3. •

Remark 3.1. If the condition $\int_0^\infty x\sigma(x)\,dx = \infty$ is satisfied, the proof of Theorem 1.3 not only shows that the support of u stays bounded for all time, but also gives a (more or less explicit) bound: $\operatorname{supp} u(t) \subset \{x \in \mathbb{R} : x \leq g(t)\}$, where the function g is given by (3.8).

4 Radial symmetry in two dimensions

Theorem 1.6 is proved by comparison with radially symmetric solutions of the same problem. Let v be a radially symmetric solution of Problem (P). Then

$$\rho v_t = \frac{1}{r}(rA(v)_r)_r \quad \text{for } 0 < r < \infty,\ t > 0.$$

By the change of variables $s = \log r$ we find

$$\hat\rho(s) v_t = A(v)_{ss} \quad \text{for } -\infty < s < \infty,\ t > 0,$$

where $\hat\rho(s) = r^2\rho(r)$. Note that $\hat\sigma(s) := \min_{0\le\xi\le s}\hat\rho(\xi) = \sigma(r)$. Theorem 1.3 states that the behaviour of interfaces depends on the integrability of $s\hat\rho(s)$ and $s\hat\sigma(s)$ at infinity. This translates in the following way:

$$\int_0^\infty s\hat\rho(s)\,ds < \infty \iff \int_1^\infty \rho(r) r \log r\,dr < \infty$$

and

$$\int_0^\infty s\hat\sigma(s)\,ds = \infty \iff \int_1^\infty \sigma(r)\frac{\log r}{r}\,dr = \infty.$$

The statement of Theorem 1.6 then follows from Theorem 1.3. Note that the extra condition $0 \in \operatorname{Int}(\operatorname{supp} u_0)$ guarantees that we can find a subsolution with non-trivial support. •

The result of Theorem 1.6 is made possible by the existence of a scaling of the independent variable r ($s = \log r$) that maps the point $r = \infty$ to $s = \infty$ and gives the equation a one-dimensional form. This same scaling maps the point $r = 0$ to $s = -\infty$, which implies that by following exactly the same reasoning we can prove

Theorem 4.1 *Let u be a solution of Problem (P) with initial condition u_0, let $\rho(x) = \rho(|x|)$, and suppose that $0 \notin \operatorname{supp} u_0$.*

(i) If $\int_0^1 \rho(r) r \log r\,dr < \infty$, then after finite time $\operatorname{supp} u(t)$ shall contain the point $x = 0$;

(ii) If $\int_0^1 \sigma(r)\frac{\log r}{r}\,dr = \infty$, then $0 \notin \operatorname{supp} u(t)$ for all time $t > 0$.

Example. In [1] the authors describe a so-called *focusing solution* of the N-dimensional porous medium equation

$$u_t = \Delta u^m \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+. \tag{4.1}$$

The support of this solution contains a hole that shrinks as time increases, disappearing totally at some finite time t^*. The solution that they construct is radially symmetric and of self-similar form: if we set $t^* = 0$, and let v denote the (scaled) pressure associated with (4.1), $v = mu^{m-1}/(m-1)$, then the solution is given by

$$v(r,t) = r^{2-\alpha}\frac{\varphi(\eta)}{-\eta}, \quad r > 0,\ t < 0,$$

where the self-similar variable η is given by $\eta = tr^{-\alpha}$. The function φ and the exponent $\alpha \in (1,2)$ are obtained by solving the ensuing ordinary differential equation.

In the case $N = 2$ we can use this solution to construct an explicit example of disappearing interfaces. Again we perform the change of variables $s = \log r$, after which the solution u given by Aronson and Graveleau satisfies the equation

$$\hat{\rho}(s)u_t = (u^m)_{ss} \quad \text{on } \mathbb{R},$$

where $\hat{\rho}(s) = e^{2s}$. Initially—that is, at some finite time before $t = 0$—$\operatorname{supp} u = [-a, \infty)$, where a is a positive number. The transformation $s = \log r$ maps $r = 0$ to $s = -\infty$, and the closure of the hole in the support in the original variables therefore corresponds to a disappearing of the left interface, clearly in finite time. Given the results of this paper, this also follows directly from the form of $\hat{\rho}$. The interest of this solution lies in the fact that the interface is given *explicitly*. The location of the interface is given by $r = c(-t)^{1/\alpha}$ in the original variables; in terms of s and t, the interface lies at

$$s = \frac{1}{\alpha}\log(-t) + c', \quad t < 0.$$

Appendix A. Well-posedness and a priori estimates

This appendix is devoted to the proofs of existence and uniqueness of the solution of the Cauchy Problem

$$(\mathrm{P}) \begin{cases} \rho(x)u_t = \Delta A(u) & \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^N \end{cases}$$

in one and two space dimensions. We can write problem (P) in the equivalent form

$$(\mathrm{P}_\beta) \begin{cases} \rho(x)\beta(v)_t = \Delta v & \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ v(x,0) = A(u_0(x)) & \text{for } x \in \mathbb{R}^N \end{cases}$$

where $v = A(u)$ and $\beta = A^{-1}$.

We borrow the definition of a weak solution from [3]. Set $Q = \mathbb{R}^N \times \mathbb{R}^+$, and $Q_T = \{(x,t) \in Q : t \le T\}$.

Definition A.1 *The function $u \in C(\bar{Q})$ is a weak solution of Problem (P) if*

(i) $0 \le u \le 1$ on $\bar{Q}$;

(ii) u satisfies the integral identity

$$(\mathrm{A.1}) \quad \int_\Omega \rho(x)u(x,t)\psi(x,t)\,dx - \int_\Omega \rho(x)u_0(x)\psi(x,0)\,dx = \int_0^t\int_\Omega \{\rho u\psi_t + A(u)\Delta\psi\}\,dxd\tau - \int_0^t\int_{\partial\Omega} A(u)\frac{\partial\psi}{\partial\nu}\,dxd\tau$$

for all smooth bounded domains $\Omega \subset \mathbb{R}^N$, for all non-negative functions $\psi \in C^{2,1}(\bar{\Omega} \times [0,T])$ that vanish on $\partial\Omega$ for all $t > 0$.

Weak sub- and supersolutions are defined similarly, after replacement in (A.1) of the equality sign by '$\le$' (for subsolutions) or '$\ge$' (for supersolutions).

We establish the following result.

Theorem A.2 *Let N be equal to either one or two. There exists a weak solution of Problem (P).*

Proof. We prove the theorem for $N = 2$, the extension to $N = 1$ being straightforward. We set $\Omega_n = \{x \in \mathbb{R}^2 : |x| < n\}$ and $Q_{nT} = \Omega_n \times (0, T)$ and we consider the problem

$$(\mathrm{A.2}) \qquad (\mathrm{P}_n) \begin{cases} \rho_n \beta_n(v)_t = \Delta v & (x,t) \in Q_{nT} \\ \dfrac{\partial v}{\partial \nu} = 0 & (x,t) \in \partial\Omega_n \times (0,T) \\ v(x,0) = v_{0n}(x) & x \in \Omega_n \end{cases}$$

in which

(i) $\rho_n \in C^\infty(\Omega_n)$, $\rho_n > 0$, and $\rho_n \to \rho$ pointwise in $\mathbb{R}^2$;

(ii) $\beta_n \in C^\infty([0, A(1)])$, $\beta_n' \geq b_0 > 0$ on $[0, A(1)]$, $\beta_n \to \beta$ uniformly on $[0, A(1)]$, and $\beta_n' \to \beta'$ in $L^1(0, A(1))$;

(iii) $v_0 = A(u_0)$; $v_{0n} \in C^\infty(\Omega_n)$, $1/n \leq v_{0n} \leq A(1) - 1/n$, and $v_{0n} \to v_0$ almost everywhere on $\mathbb{R}^2$.

Problem (P_n) has a unique classical solution v_n [11] and it follows from the comparison principle that $1/n \leq v_n \leq A(1) - 1/n$ on Q_{nT}.

We conclude from [4] that there exists a function $v \in C(\bar{Q})$ and a subsequence $\{v_{n_k}\}$ such that $v_{n_k} \to v$ uniformly on $\{|x| \leq R\} \times [0,T]$ for all R. We deduce from a similar identity for v_n that v satisfies the integral identity

$$\int_\Omega \rho(x)\beta(v(x,t))\psi(x,t)\,dx - \int_\Omega \rho(x)u_0(x)\psi(x,0)\,dx =$$
$$\int_0^t \int_\Omega \{\rho\beta(v)\psi_t + v\Delta\psi\}\,dxd\tau - \int_0^t \int_{\partial\Omega} v\frac{\partial\psi}{\partial\nu}\,dxd\tau$$

for all smooth bounded domains $\Omega \subset \mathbb{R}^2$, for all functions $\psi \in C^{2,1}(\bar{\Omega} \times [0,T])$ which vanish on $\partial\Omega$ and for all $t > 0$. The function $u = \beta(v)$ satisfies the assertion of the theorem. •

The proof of Theorem 1.1 that we give here is an adaptation of the proof of a similar property due to Bertsch, Kersner, and L. A. Peletier [3]. It should be noted that although the techniques are similar, there is an interesting effect in the change from one or two spatial dimensions to three dimensions and higher. This is further explained in Remark A.1.

Proof of Theorem 1.1. Again we only prove the theorem for $N = 2$; the extension to $N = 1$ is straightforward.

Define the functions $w = u_1 - u_2$ and $w_0 = u_{01} - u_{02}$. They satisfy

$$(\mathrm{A.3}) \qquad \int_\Omega \rho w(\cdot,t)\psi(\cdot,t) - \int_\Omega \rho w_0 \psi(\cdot,0) \leq$$
$$\leq \int_0^t \int_\Omega (w\rho\psi_t + (A(u_1) - A(u_2))\Delta\psi) - \int_0^t \int_{\partial\Omega} (A(u_1) - A(u_2))\psi_\nu$$

for all appropriate domains Ω and test functions ψ. For the length of this proof we adopt the notation $\psi_\nu = \partial\psi/\partial\nu$. Define $\Omega_n = \{x \in \mathbb{R}^2 : |x| < n\}$ and $Q_{nt} = \Omega_n \times (0,t]$, and the function

$$q(x,t) = \begin{cases} \dfrac{A(u_1) - A(u_2)}{u_1 - u_2} & \text{if } u_1 \neq u_2 \\ 0 & \text{if } u_1 = u_2 \end{cases}$$

Remark that $q \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^+)$, and that $\|q\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^+)} \leq \|A'\|_{L^\infty(0,1)}$. We approximate q on Q_{nt} by functions q_n such that

(A.4) $$n^{-2} \leq q_n \leq \|q\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^+)} + n^{-2} \text{ on } Q_{nt};$$

(A.5) $$\|(q_n - q)/\sqrt{q_n}\|_{L^2(Q_{nt})} \to 0 \text{ as } n \to \infty,$$

and introduce as test functions the solutions ψ_n of

(A.6) $$\begin{cases} \rho\psi_t + q_n\Delta\psi = 0 & \text{in } Q_{nt} \\ \psi = 0 & \text{on } \partial\Omega_n \times [0,t] \\ \psi(x,t) = \chi(x) & \text{on } \Omega_n, \end{cases}$$

where χ is a fixed function that belongs to $C_c^\infty(\Omega_n)$ for n large enough and takes values in $[0,1]$. The density ρ is bounded from below on Q_{nt}, so (A.6) has a unique solution $\psi_n \in C^{2,1}(\bar{Q}_{nt})$. By multiplying the equation for ψ with $\Delta\psi/\rho$ we find that

(A.7) $$\int_0^t \int_{\Omega_n} q_n(\Delta\psi)^2 \leq C$$

where C is a constant independent of n.

Using ψ_n as a test function in (A.3) we find that

$$\int_{\Omega_n} \rho\chi w(\cdot,t)\,dx - \int_{\Omega_n} \rho w_0\psi_n(\cdot,0)\,dx \leq \int_0^t \int_{\Omega_n} (q - q_n)\Delta\psi_n - \int_0^t \int_{\partial\Omega_n} qw\psi_{n\nu}$$

Denote the two integrals on the right-hand side I_1 and I_2. We shall now show that both tend to zero as n tends to infinity. First consider I_1:

$$I_1^2 \leq \int_0^t \int_{\Omega_n} \left|\frac{q - q_n}{\sqrt{q_n}}\right|^2 \int_0^t \int_{\Omega_n} q_n|\Delta\psi_n|^2$$

and the right-hand side of this expression tends to zero because of (A.7) and (A.5). To prove that I_2 tends to zero, we compare the function ψ_n with the solution z_n of

$$\begin{cases} \Delta z = 0 & r_0 < |x| < n \\ z = 0 & |x| = n \\ z = 1 & |x| = r_0 \end{cases}$$

where r_0 is such that $\operatorname{supp}\chi \subset \{|x| \leq r_0\}$. The solution z_n of this problem is $z_n(x) = (\log n - \log|x|)/(\log n - \log r_0)$. Since both ψ_n and z_n are equal to zero on $|x| = n$, we have

$$0 \leq -\psi_{n\nu} \leq -z_{n_\nu} \quad \text{on } \partial\Omega_n.$$

Explicitly this implies that

(A.8) $$|\psi_{n\nu}| \le \frac{1}{n(\log n - \log r_0)}.$$

We can then estimate I_2 by

$$|I_2| \le t\|A'\|_{L^\infty(0,1)} \frac{2\pi}{\log n - \log r_0}$$

and the right-hand side of this expression tends to zero as n tends to infinity.

Since by the comparison principle $0 \le \psi_n \le 1$ on Q_{nT}, we can deduce from (4) that

(A.9) $$\begin{aligned}\int_{\mathbb{R}^2} \rho\chi w(\cdot,t)\,dx &\le I_1 + I_2 + \int_{\Omega_n} \rho w_0 \psi_n(\cdot,0)\,dx \\ &\le I_1 + I_2 + \int_{\mathbb{R}^2} \rho w_{0+}\,dx.\end{aligned}$$

The right-hand side of this expression is finite by the hypothesis of the Theorem. Passing to the limit in (A.9) yields

$$\int_{\mathbb{R}^2} \rho\chi w(\cdot,t)\,dx \le \int_{\mathbb{R}^2} \rho w_{0+}\,dx$$

for all $\chi \in C_c^\infty(\mathbb{R}^2)$ such that $0 \le \chi \le 1$. The theorem then follows immediately from this inequality by letting χ converge pointwise to the function $\operatorname{sgn}(w_+)$. •

Remark A.1. The absence of a uniform lower bound for ρ introduces an interesting effect in the well-posedness of the Cauchy Problem for equation (1.1). If the proof of Theorem 1.1 is rewritten for spatial dimensions different from $N = 2$, the only important difference lies in the explicit function z_n. In one dimension, $z_n(x) = (n-x)/(n-r_0)$, so that $z_n'(n) = -1/(n-r_0)$ tends to zero as $n \to \infty$. This implies that I_2 tends to zero as $n \to \infty$, which is necessary to conclude. However, when $N \ge 3$, $z_n(r) = (r^{2-N} - n^{2-N})/(r_0^{2-N} - n^{2-N})$. In this case, $\int_{\partial\Omega_n} |z_n'|$ remains bounded away from zero, and without an additional assumption on the solution in fact uniqueness does not hold ([9], [5], [6]).

Remark A.2. The proof of the comparison principle still holds when the condition $A \in C^1([0,1])$ is replaced by $A \in W^{1,\infty}(0,1)$ and the condition $u_0 \in C(\mathbb{R}^N)$, $0 \le u_0 \le 1$ by $u_0 \in L^\infty(\mathbb{R}^N)$, $0 \le u_0 \le 1$ a.e. on $\mathbb{R}^N$.

We conclude this appendix with the proof of Lemma 2.1.

Proof of Lemma 2.1. We first prove the second part of the Lemma. By Theorems A.2 and 1.1 we can obtain v as the limit of functions v_n, which are defined for all $|x| < n$ and $0 \le t \le \tau$. First fix $R > 0$ and set $B_R = \{x \in \mathbb{R}^N : |x| < R\}$. We multiply the differential equation in Problem (P_n) by v_n and integrate on $\{|x| < n\} \times (0,\tau)$:

(A.10) $$\begin{aligned}&\int_{B_R} \rho_n(x) B_n(v_n(x,\tau))\,dx + \int_0^\tau \int_{B_R} |\nabla v_n|^2 \\ &\le \int_{|x|<n} \rho_n(x) B_n(v_n(x,\tau))\,dx + \int_0^\tau \int_{|x|<n} |\nabla v_n|^2 \\ &= \int_{|x|<n} \rho_n(x) B_n(v_{0n})\,dx,\end{aligned}$$

where $B_n(s) = \int_0^s \tau\beta_n'(\tau)d\tau$. The condition $\int \rho u_0 < \infty$ implies that the functions v_{0n} can be chosen such that $\int \rho_n\beta_n(v_{0n})$ is bounded independently of n. Since the function $B_n \circ \beta_n^{-1}$ is Lipschitz continuous with a Lipschitz constant L that does not depend on n, the last term in (A.10) is bounded as $n \to \infty$ and therefore we can extract a subsequence—without changing notation—such that ∇v_n converges weakly in $L^2(B_R \times (0,\tau))$. With the uniform convergence of v_n we can identify the limit as ∇v. Using the dominated convergence theorem and the weak convergence of ∇v_n we can pass to the limit in (A.10) to obtain

$$\int_{B_R} \rho(x)B(v(x,\tau))\,dx + \int_0^\tau \int_{B_R} |\nabla v|^2 \le \int_{\mathbb{R}^N} \rho(x)B(v_0)\,dx.$$

The result then follows from the monotone convergence theorem.

To prove part (i), consider a monotonic cut-off function $\eta \in C^\infty(\mathbb{R})$ such that $\eta = 1$ on $(-\infty, 1]$ and $\eta = 0$ on $[2,\infty)$. Take $\psi(x) = \eta(|x|/R)$ for some $R > 0$ as a test function in (A.1), giving

$$\int_{\mathbb{R}^N} \rho u(\cdot,\tau)\psi = \int_{\mathbb{R}^N} \rho u_0\psi - \int_0^\tau \int_{\mathbb{R}^N} \nabla v \nabla\psi \tag{A.11}$$

where we have used the fact that $\nabla v \in L^2(\mathbb{R}^N \times (0,\tau))$ by part (ii). We can estimate the last integral in (A.11) by

$$R^{N/2-1} \max_{\mathbb{R}} |\eta'| \left(\int_0^\tau \int_{R<|x|<2R} |\nabla v|^2 \right)^{1/2}$$

which tends to zero as $R \to \infty$. The result then follows from an application of the monotone convergence theorem.

To prove part (iii), multiply by tv_{nt} the equation satisfied by v_n and integrate:

$$\begin{aligned}\int_0^\tau \int_{|x|<n} t\rho_n\beta_n'(v_n)v_{nt}^2 &= -\frac{1}{2}\int_0^\tau \int_{|x|<n} t\frac{d}{dt}|\nabla v_n|^2 \\ &= \frac{1}{2}\int_0^\tau \int_{|x|<n} |\nabla v_n|^2 - \frac{\tau}{2}\int_{|x|<n} |\nabla v_n|^2(\cdot,\tau),\end{aligned}$$

or

$$\tau \int_{|x|<n} |\nabla v_n|^2(\cdot,\tau) \le \int_0^\tau \int_{|x|<n} |\nabla v_n|^2,$$

after which the result follows from the second part of the Lemma. •

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